Integrable Chain Model with Additional Staggered Model Parameter

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Abstract

The generalization of the Yang-Baxter equations (YBE) in the presence of $\mathbb{Z}_2$ grading along both chain and time directions is presented. The XXZ model with staggered disposition along a chain of both, the anisotropy $\pm\Delta$, as well as shifts of the spectral parameters are considered and the corresponding integrable model is constructed. The Hamiltonian of the model is computed in fermionic and spin formulations. It involves three neighbour site interactions and therefore can be considered as a zig-zag ladder model. The Algebraic Bethe Ansatz technique is applied and the eigenstates, along with eigenvalues of the transfer matrix of the model are found. The model has a free fermionic limit at $\Delta = 0$ and the integrable boundary terms are found in this case.

This construction is quite general and can be applied to other known integrable models.

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1 Introduction

The construction of integrable chain models with inhomogeneous model parameters is very important in many aspects. Besides its own interest in mathematical physics, its possible applications in the problems of condensed matter physics is hard to miss-estimate. Even more, we believe that in order to proceed further in the non-critical string theory and cross the so called $c = 1$ [1] barrier, it is necessary to consider strongly correlated fields on a line, instead of the free field degrees of freedom on the world sheet of the string (as in $d \leq 1$ case) .

The crucial problem for strings in $d \geq 1$ is the problem of the tachyon, which means a non-trivial structure of the ground state. Therefore, the investigation of strongly correlated chain systems, which can be considered on a fluctuating line, is actual.

In this sense, in order to be able to take into account non-trivial ground state correctly, it is straightforward to consider an integrable $2d$ model on a fluctuating surface. One attractive possibility is the consideration of inhomogeneously shifted spectral parameters in R-matrices along a chain. It is tempting to consider first the case of staggered inhomogeneity.

An interesting model of hopping fermions in the Manhattan lattice, where the hopping parameters are staggered in both time and chain directions, appeared in connection with three dimensional Ising Model [2, 3]. In a particular case of hopping parameters, this produces a $N \times N$ numerical transfer matrix [4] in connection with edge excitations in the Hall effect.

Since the pioneering work of H. Bethe [5] and the essential input made by C. Yang [6], the theory of $2d$-integrable models is well developed presently and has its formulation in technique, called Algebraic Bethe Ansatz (ABA) (or Quantum Inverse Scattering Method(QISM)) [7, 8].

The basic element of this method is the matrix $R_{ij}(\lambda, u)$ (λ is the model parameter and u is the spectral parameter), the product of which along a chain defines the monodromy matrix $T(\lambda, u)$ as follows

$$T(\lambda, u) = \prod_{j=1}^{N} R_{0j}(\lambda, u). \tag{1}$$

The trace of $T(\lambda, u)$ over states along the chain defines the transfer matrix $\tau(\lambda, u)$, the L's power of which (L is the lattice size in time direction) is the partition function of the model.

When any two transfer matrices of the model with different spectral parameters commute then we have an infinite number of conservation currents, which causes the integrability. The sufficient condition of commutativity of the transfer matrices is the Yang-Baxter equation (YBE) for the R-matrix, any solution of which defines an integrable model.

Usually the integrable models under consideration are homogeneous along a chain, namely, the spectral parameters u in the product (1) are equal. There were attempts to

construct inhomogeneous models, for example we can mention the successful attempt in the article [9] of the model with alternating spin-$\frac{1}{2}$, spin-1 chain, but the main problem here is to obtain a local Hamiltonian (local conservation laws). It is obvious that, with the shift of the spectral parameter u in the product of $R_{0j}(\lambda, u)$ matrices at the site j by arbitrary values z_j, we still fulfill the YBE (and hence the commutativity of transfer matrices) however the Hamiltonian of the model, and the other conservation quantities, will be local only if the shifts have periodic structure. Due to periodicity which goes far from neighbour sites, the local Hamiltonian will contain interactions between sites within the period and therefore can be represented as multi-leg ladder model. This type of ladder models differs from ones, constructed in [20] by extension of the symmetry algebra, in [21] by construction first of the ground state, in [22] by use of higher order conservation laws and others [23, 24].

The aim of the present article is to construct an integrable model with commuting transfer matrices for different spectral parameters, where the R-matrices in the product (1) are staggered by their construction as well as by shift of the spectral parameter u and where the Hamiltonian (and the other conservation laws) is local. The shift of the spectral parameter produces an additional model parameter θ. Similar model, where only the shift of spectral parameter was considered (no change of the structure of the R-matrix), was constructed earlier in [10, 11].

We have generalized the YBE for that purpose in order to ensure the commutativity of the transfer matrices and demonstrate that they do have a solution in a case when the basic R-matrix is the one of the integrable XXZ-model.

In Section 2 the generalized YBE is formulated and the solution is found for the basic XXZ-model.

In Section 3 we calculate the Hamiltonian for the XXZ model in both, fermionic and spin formulations, and demonstrate that it is local. Because of the invariance of the model under double space translations, the Hamiltonian contains interaction between three neighbouring points (instead of nearest-neighbour links in the ordinary XXZ). It can hence be represented as a zig-zag ladder Hamiltonian, where we have two terms, corresponding to $SU(1,1)$ Heisenberg models on each of two chains, and the interaction between chains terms of topological form, which are written on the triangles. In the free fermionic limit ($\lambda = \pi/2$) we have obtained two independent sets of fermions, hopping only at the odd or even sites.

This method, as one in [10, 11], provides a way of construction of the integrable ladder models.

Algebraic Bethe Ansatz (ABA) technique is applied to present model in Section 4 in order to find the eigenstates and eigenvalues of the transfer matrix.

In Section 5 the open chain problem is considered and integrable reflection $\mathcal{K}$-matrices are found for the staggered XX-model. We also write the corresponding boundary terms in the Hamiltonian.

We think that our approach for the construction of integrable models with staggered

model parameters is quite general and can be applied to other integrable homogeneous models.

2 The definition of the YBE for the staggered model and its solution for the XXZ chain

The key of integrability of the models is the YBE, which implies some restrictions on the $R_{ij}(\lambda, u)$-matrix, basic constituent of the monodromy matrix (1). The YBE ensures a local sufficient condition for the commutativity of the transfer-matrices $\tau(u) = trT(u)$ at different values of the spectral parameter u, which corresponds to the rapidity of pseudoparticles of the model

$$[\tau(u), \tau(v)] = 0. \tag{2}$$

We use in the future the fermionization technique (alternative to Jordan-Wigner transformation), developed in [12, 13, 14] (see also [15]) and work with the R-operators (rather than matrices), expressed in Fermi-fields c_i, c_i^+, ($i = 1, ...N$ is the chain site).

By definition R_{aj} acts as a intertwining operator on the space of direct product of the so called auxiliary $V_a(v)$ and quantum $V_j(u)$ spaces

$$R_{aj}(u, v): \quad V_a(u) \otimes V_j(v) \to V_j(v) \otimes V_a(u) \tag{3}$$

and can be represented graphically as

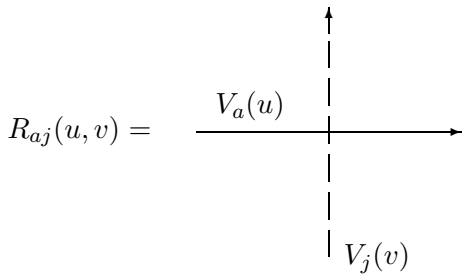

Figure 1: R_{aj} matrix

The spaces $V_a(u)$ and $V_j(v)$ with spectral parameters u and v are irreducible representations of the affine quantum group $U_q\widehat{g}$, which is the symmetry group of the integrable

model under consideration. Provided that the states $\mid a\rangle \in V_a$ and $\mid j\rangle \in V_j$ form a basis for the spaces V_a and V_j, following [12] we can represent the action of the operator R_{aj} as

$$R_{aj} \mid j_1\rangle \otimes \mid a_1\rangle = (R_{aj})^{a_2 j_2}_{a_1 j_1} \mid a_2\rangle \otimes \mid j_2\rangle, \tag{4}$$

where the summation is over the repeating indices a_2 and j_2 (but not over a_1 and j_1).

By introducing the Hubbard operators

$$X^{a_1}_{a_2} = \mid a_2\rangle\langle a_1 \mid, \qquad X^{j_1}_{j_2} = \mid j_2\rangle\langle j_1 \mid \tag{5}$$

in the graded spaces V_a and V_j correspondingly, one can rewrite (4) as

$$\begin{aligned} R_{aj} = R_{aj} \mid j_1\rangle \mid a_1\rangle\langle a_1 \mid \langle j_1 \mid &= (R_{aj})^{a_2 j_2}_{a_1 j_1} \mid a_2\rangle \mid j_2\rangle\langle a_1 \mid \langle j_1 \mid \\ &= (-1)^{p(a_1)p(j_2)} (R_{aj})^{a_2 j_2}_{a_1 j_1} X^{a_1}_{a_2} X^{j_1}_{j_2} \end{aligned} \tag{6}$$

where the sign factor takes into account the possible grading of the states $\mid a_i\rangle$ and $\mid j_i\rangle$, $p(a_i)$ and $p(j_i)$ denote the corresponding parities and the summation over the repeating indices.

In terms of operators R_{ij} the matrix valued YBE can be written in the following operator form

$$R_{ab}(u,v) R_{aj}(u,w) R_{bj}(v,w) = R_{bj}(v,w) R_{aj}(u,w) R_{ab}(u,v). \tag{7}$$

We use for V_0 and V_j the Fock space of the Fermi fields [5] c_i, c_i^+ with basis vectors $|0\rangle_i$ and $|1\rangle_i$, for which

$$(X_i)^{a'}_{a} = \begin{pmatrix} 1-n_i & c_i^+ \\ c_i & n_i \end{pmatrix}, \tag{8}$$

we have a fermionization of the model, which is equivalent to Jordan-Wigner transformation.

In case of XXZ model, the fermionic expression of the operator $R_{ij}(u)$ is easy to obtain by use of formulas (6) and (8), and the standard expression for the $R^{b'a'}_{ba}(u)$. As a result one obtains

$$\begin{aligned} R_{jk}(u) &= a(u)\left[-n_j n_k + (1-n_j)(1-n_k)\right] + b(u)\left[n_j(1-n_k) + (1-n_j)n_k\right] \\ &+ c(u)\left[c_j^+ c_k + c_k^+ c_j\right]. \end{aligned} \tag{9}$$

Let us now consider $\mathbb{Z}_2$ graded quantum $V_{j,\rho}(v)$ and auxiliary $V_{a,\sigma}(u)$ spaces, $\rho, \sigma = 0, 1$. In this case we will have 4×4 R-matrices, which act on the direct product of the spaces $V_{a,\sigma}(u)$ and $V_{j,\rho}(v)$, $(\sigma, \rho = 0, 1)$, mapping them on the intertwined direct product of $V_{a,\bar{\sigma}}(u)$ and $V_{j,\bar{\rho}(v)}$ with the complementary $\bar{\sigma} = (1-\sigma)$, $\bar{\rho} = (1-\rho)$ indices

$$R_{aj,\sigma\rho}(u,v): \quad V_{a,\sigma}(u) \otimes V_{j,\rho}(v) \to V_{j,\bar{\rho}}(v) \otimes V_{a,\bar{\sigma}}(u). \tag{10}$$

[5] if the dimensions of the spases V_0 and V_j are more than two, then one can use Fock spaces of more fermions, see [13]

It is convenient to introduce two transmutation operations ι_1 and ι_2 with the property $\iota_1^2 = \iota_2^2 = id$ for the quantum and auxiliary spaces correspondingly, and to mark the operators $R_{aj,\sigma\rho}$ as follows

$$\begin{aligned} R_{aj,00} &\equiv R_{aj}, \qquad R_{aj,01} \equiv R^{\iota_1}_{aj}, \\ R_{aj,10} &\equiv R^{\iota_2}_{aj}, \qquad R_{aj,11} \equiv R^{\iota_1\iota_2}_{aj}. \end{aligned} \tag{11}$$

The introduction of the $\mathbb{Z}_2$ grading in quantum space means, that we have now two monodromy matrices $T_\rho, \rho = 0, 1$, which act on the space $V_\rho = \prod_{j=1}^N V_{j,\rho}$ by mapping it on $V_{\bar\rho} = \prod_{j=1}^N V_{j,\bar\rho}$

$$T_\rho \qquad : V_\rho \to V_{\bar\rho}, \qquad\qquad \rho = 0, 1. \tag{12}$$

It is clear now, that the monodromy matrix of the model, which should define the partition function, is the product of two monodromy matrices

$$T(\lambda, u) = T_0(\lambda, u) T_1(\lambda, u). \tag{13}$$

Now, because of the grading in the auxiliary space, we would like to construct the monodromy matrices $T_{0,1}$ as a staggered product of the R_{aj} and $\bar R^{\iota_2}_{aj}$ matrices. Let us define

$$\begin{aligned} T_1(\lambda, u) &= \prod_{j=1}^N R_{a,2j-1}(\lambda, u) \bar R^{\iota_2}_{a,2j}(\lambda, u) \\ T_0(\lambda, u) &= \prod_{j=1}^N \bar R^{\iota_1}_{a,2j-1}(\lambda, u) R^{\iota_1\iota_2}_{a,2j}(\lambda, u), \end{aligned} \tag{14}$$

where the notation $\bar R$ in general means the different parameterization of the R-matrix via models λ and u parameters and can be considered as an operation over R with property $\bar{\bar R} = R$.

Graphically the formulas (14) can be expressed as in figure 2.

In order to have a integrable model with commuting transfer matrices (13) for different spectral parameters

$$[trT(\lambda, u),\ trT(\lambda, v)] = 0 \tag{15}$$

it is enough to have the following relations for the $\tau_\sigma(u) = trT_\sigma(\lambda, u), \qquad (\sigma = 0, 1)$

$$\tau_\sigma(\lambda, u)\tau_{1-\sigma}(\lambda, v) = \bar\tau_\sigma(\lambda, v)\bar\tau_{1-\sigma}(\lambda, u), \qquad \sigma = 0, 1 \tag{16}$$

It is not hard to see, that in order to ensure the commutativity condition (15) the R- and $\bar R$-matrices in the expression (14) should fulfill the following two Yang-Baxter equations

$$R_{ab}(u, v)\bar R^{\iota_1}_{a,j}(\lambda, u) R_{b,j}(\lambda, v) = R^{\iota_1}_{b,j}(\lambda, v)\bar R_{a,j}(\lambda, u)\tilde R_{ab}(u, v) \tag{17}$$

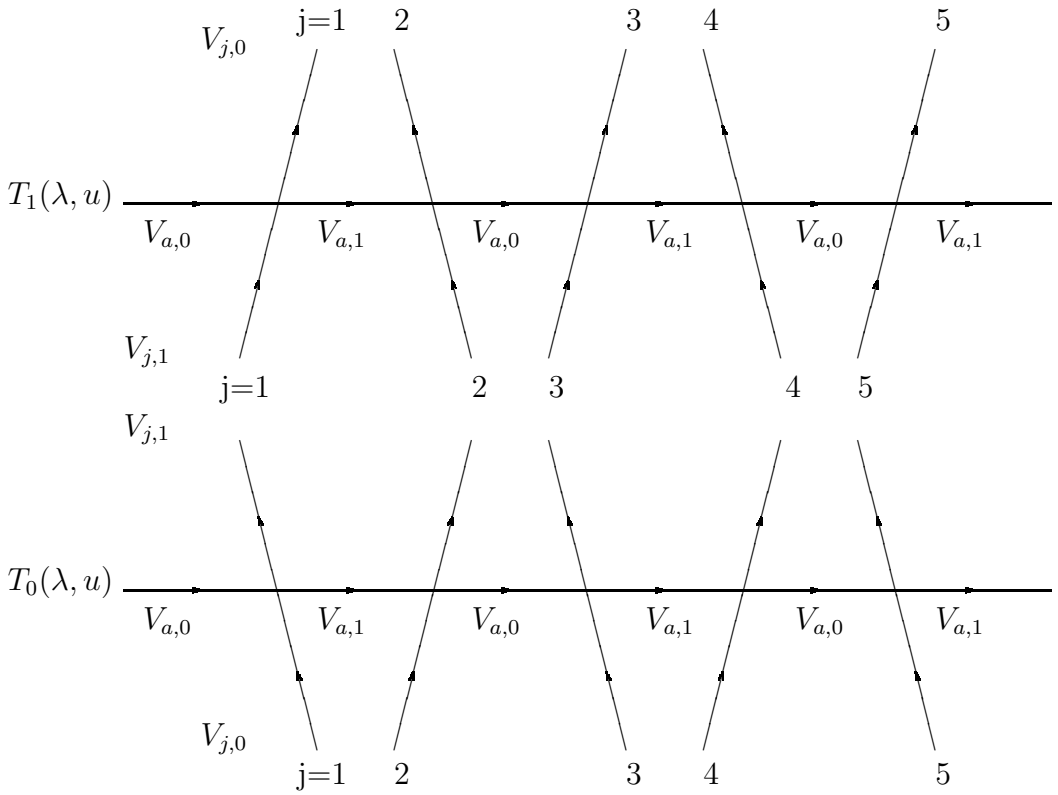

Figure 2: Monodromy matrices T_0 and T_1

and

$$\tilde{R}_{ab}(u,v)R^{\iota_1\iota_2}_{a,j+1}(\lambda,u)\bar{R}^{\iota_2}_{b,j+1}(\lambda,v) = \bar{R}^{\iota_1\iota_2}_{b,j+1}(\lambda,v)R^{\iota_2}_{a,j+1}(\lambda,u)R_{ab}(u,v). \tag{18}$$

By inserting the expression (9) for the R-operator into (17) one can obtain 20 equations on parameters a, b, c and $\bar{a}$, $\bar{b}$, $\bar{c}$, which after defining the transposition operation ι_1 and $\sim$ as

$$\begin{aligned} a^{\iota_1}(u) &= a(u), \qquad b^{\iota_1}(u) = -b(u), \qquad c^{\iota_1} = c(u) \\ \tilde{a}(u,v) &= a(u,v), \qquad \tilde{b}(u,v) = -b(u,v), \qquad \tilde{c}(u,v) = c(u,v), \end{aligned} \tag{19}$$

are reducing to following three equations

$$\begin{aligned} a(u,v)\bar{c}(u)a(v) &= c(u,v)\bar{a}(u)c(v) + b(u,v)\bar{c}(u)b(v), \\ b(u,v)\bar{a}(u)c(v) + c(u,v)\bar{c}(u)b(v) &= a(u,v)\bar{b}(u)c(v), \\ c(u,v)\bar{b}(u)a(v) &= b(u,v)\bar{c}(u)c(v) + c(u,v)\bar{a}(u)b(v). \end{aligned} \tag{20}$$

It is easy to recognize in equations (20) the ordinary YBE for the XXZ model, where we suppose that the parametrization of the variables $\bar{a}(u)$, $\bar{b}(u)$, $\bar{c}(u)$ differs from $a(u)$, $b(u)$, $c(u)$. We are going to introduce a new model parameter here for them.

The second set of YBEs (18) gives

$$
\begin{aligned}
a(u,v)c^{\iota_2}(u)\bar{a}^{\iota_2}(v) &= c(u,v)a^{\iota_2}(u)\bar{c}^{\iota_2}(v) - b(u,v)c^{\iota_2}(u)\bar{b}^{\iota_2}(v),\\
-b(u,v)a^{\iota_2}(u)\bar{c}^{\iota_2}(v) &+ c(u,v)c^{\iota_2}(u)\bar{b}^{\iota_2}(v) = a(u,v)b^{\iota_2}(u)\bar{c}^{\iota_2}(v),\\
c(u,v)b^{\iota_2}(u)\bar{a}^{\iota_2}(v) &= -b(u,v)c^{\iota_2}(u)\bar{c}^{\iota_2}(v) + c(u,v)a^{\iota_2}(u)\bar{b}^{\iota_2}(v).
\end{aligned} \tag{21}
$$

which differs from the first set (20) by the transposition ι_2 of the variables, by displaced bars from the $\bar{a}(u)$, $\bar{b}(u)$, $\bar{c}(u)$ on to $a(v)$, $b(v)$, $c(v)$ and change of sign $b(u,v) \to -b(u,v)$. This last change of the sign comes from the fact that, in the first set of YBEs (17), the variables $\tilde{b}(u,v)$ stationed in the right side of equations, whilst in the second set (18) - in the left side.

The solution of the equations (20) is the same as for ordinary XXZ, but we will make a different shifts of the spectral parameters for the $\bar{a}(u)$, $\bar{b}(u)$, $\bar{c}(u)$ and $a(u)$, $b(u)$, $c(u)$.

Namely, if we take

$$
\begin{aligned}
a(u) &= \sin(\lambda+u), \qquad \bar{a}(u) = a(\bar{u}) = \sin(\lambda+\theta-u),\\
b(u) &= \sin(u), \qquad\quad \bar{b}(u) = b(\bar{u}) = \sin(\theta-u),\\
c(u) &= \sin(\lambda), \qquad\quad \bar{c}(u) = c(\bar{u}) = \sin(\lambda),
\end{aligned} \tag{22}
$$

then the solution of (20) for $a(u,v)$, $b(u,v)$, $c(u,v)$ is

$$
\begin{aligned}
a(u,v) &= a(\bar{u}-v) = \sin(\lambda+\bar{\eta}-u-v),\\
b(u,v) &= b(\bar{u}-v) = \sin(\bar{\eta}-u-v),\\
c(u,v) &= c(\bar{u}-v) = \sin(\lambda),
\end{aligned} \tag{23}
$$

where $\bar{u} = \theta - u$ and θ is the new parameter of the model.

It is easy to see after simple calculations, that the second set of YBEs (21) is compatible with the first set (20) by producing the same solution (23), if we define the ι_2 transposition as follows

$$
\begin{aligned}
a^{\iota_2}(u) &= \sin(\lambda-u), \qquad b^{\iota_2}(u) = b(u), \qquad c^{\iota_2} = c(u),\\
\bar{a}^{\iota_2}(u) &= \sin(\lambda-\bar{u}), \qquad \bar{b}^{\iota_2}(u) = b(\bar{u}), \qquad \bar{c}^{\iota_2} = c(\bar{u}).
\end{aligned} \tag{24}
$$

Therefore, the formula (22) together with the definitions (19) and (24) defines a solution of the double set of YBEs (17) and (18). The new parameter θ appeared in the model in addition to λ (or Δ) in the ordinary XXZ-model.

As it is well known, the variable

$$
\Delta = \frac{a^2(u) + b^2(u) - c^2(u)}{2a(u)b(u)} = \cos\lambda = const \tag{25}
$$

characterize the anisotropy of the product of the spins $\vec{\sigma}_j$ and $\vec{\sigma}_{j+1}$ in z-direction in the Heisenberg model.

It is important to mention here, that the transpositions ι_1 and ι_2 changes the sign of Δ, $\Delta_{\iota_1} = \Delta_{\iota_2} = -\Delta$. This means, that in a definition (14) of the monodromy matrices $T_{0,1}(\lambda, u)$ we have a product of R-matrices with alternating anisotropy parameters $\pm\Delta$. This fact essentially differs our construction from the one in the articles [10, 11], where the simple shift of the spectral parameter is considered.

3 The Hamiltonian of the model. Ladder form

The prove of existence and the calculation of the local Hamiltonian in our model is easy to carry out technically in a so called check-formalism, where instead of usual R-matrix, we consider the matrix $\check{R} = PR$, (P being the permutation map $P = x \otimes y \to y \otimes x$). In matrix elements

$$\check{R}_{ab}^{a'b'} = R_{ab}^{b'a'}. \tag{26}$$

Instead of the operator R_{aj} in (6) let us define

$$\check{R}_{ij} = (\check{R}_{ij})_{ab}^{a'b'} \mid a' \rangle \mid b' \rangle\langle b \mid \langle a \mid = (-1)^{p(a')(p(b')+p(b))} (\check{R}_{ij})_{ab}^{a'b'} X_{a'}^{a} X_{b'}^{b}, \tag{27}$$

which after the fermionization (8) will have the form

$$\begin{aligned} \check{R}_{jk}(u) &= a(u)\left[n_j n_k + (1-n_j)(1-n_k)\right] + c(u)\left[n_j(1-n_k) + (1-n_j)n_k\right] \\ &+ b(u)\left[c_j^+ c_k + c_k^+ c_j\right]. \end{aligned} \tag{28}$$

and where the parameters $a(u)$, $b(u)$, $c(u)$ are defined by formulas (22).

The monodromy matrix (14) in this formalism has the following expression

$$\begin{aligned} T_1(\lambda, u) &= \prod_{j=1}^{N} \check{R}_{2j,2j+1}(\lambda, u) \check{\tilde{R}}_{2j+1,2j+2}^{\iota_2}(\lambda, u), \\ T_0(\lambda, u) &= \prod_{j=1}^{N} \check{\tilde{R}}_{2j+1,2j+2}^{\iota_1}(\lambda, u) \check{R}_{2j+2,2j+3}^{\iota_1\iota_2}(\lambda, u). \end{aligned} \tag{29}$$

It is convenient to represent graphically the fermionic $\check{R}_{j,j+1}$-operators as in figure 3.

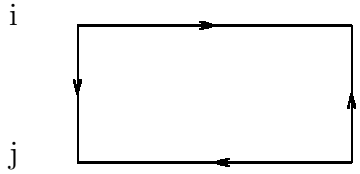

Figure 3: $\check{R}_{ij}$

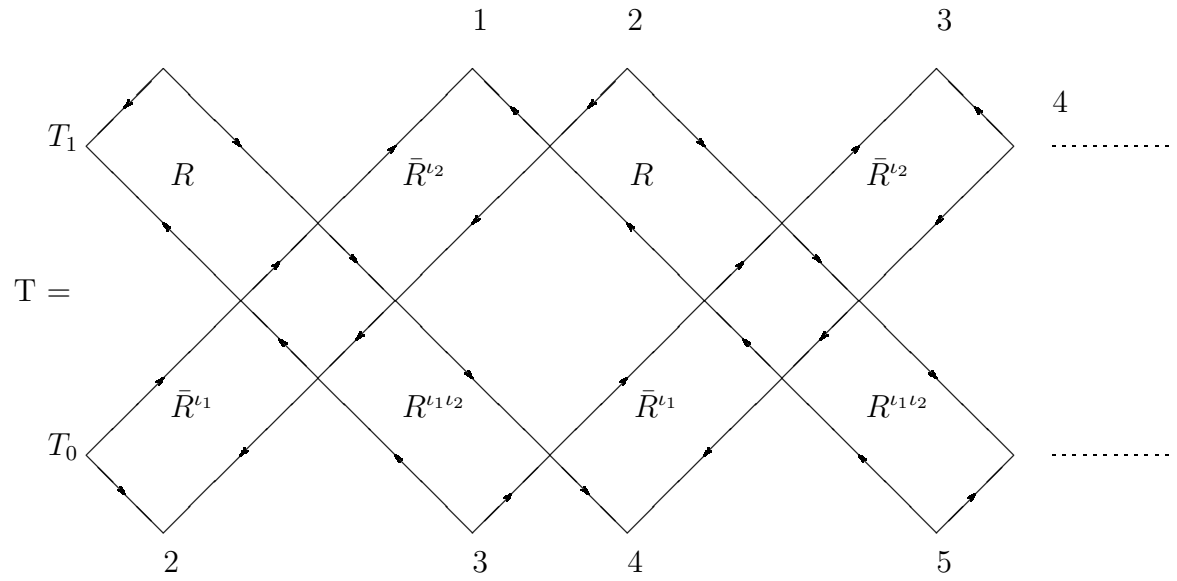

Figure 4: Monodromy matrix

Then, the full monodromy matrix $T(\lambda, u) = T_0(\lambda, u)T_1(\lambda, u)$, defined by use of formulas (29), can be pictured as in figure 4, which also demonstrates the order of writing of the operators in the product.

In order to be able to write a local Hamiltonian, it is necessary to have a value

$$\check{R}_{ij}(u_0) = I_{ij}. \tag{30}$$

Then the logarithmic derivative of the transfer matrix at this point will define a local Hamiltonian

$$H = -\frac{\partial \ln trT\,(u)}{\partial u}\,|_{u=u_0}\,. \tag{31}$$

Technically, in order to find a Hamiltonian, one should expand $\check{R}_{ij}$-operators around u_0 and extract the first order terms over $(u-u_0)$ in the product $T(\lambda, u) = T_0(\lambda, u)T_1(\lambda, u)$. In ordinary homogeneous case by inputting the first order expansions of the $\check{R}$-operators around unity into the product

$$T(\lambda, u) = \prod_{j=1}^{N} \check{R}_{j,j+1}(\lambda, u) \tag{32}$$

it is easy to see that there is no scattering between long distance fermions and only the nearest-neighbour hopping terms are contributing into the Hamiltonian.

In the present model we have a two different R-operators in the product (29), only one of them becoming unity for some value of the spectral parameter. Let us take $u_0 = 0$.

Then, by expanding around the point $u=0$ the $\check{R}, \check{R}^{\iota_2\iota_2}, \overset{\vee}{\check{R}}^{iota_1}, \overset{\vee}{\check{R}}^{\iota_2}$ operators, defined by formulas (19),(22), (24) and (28), obtains

$$\begin{aligned}
\check{R}_{2i,2i+1}(u) &= \sin\lambda I_{2i,2i+1} + u(\cos\lambda P_{2i,2i+1} + c^+_{2i}c_{2i+1} + c^+_{2i+1}c_{2i}),\\
\check{R}^{\iota_1\iota_2}_{2i,2i+1}(u) &= \sin\lambda I_{2i,2i+1} - u(\cos\lambda P_{2i,2i+1} + c^+_{i}c_{i+1} + c^+_{i+1}c_{i}),\\
\overset{\vee}{\check{R}}^{\iota_1}_{2i-1,2i}(u) &= \check{R}_{2i-1,2i}(\theta) - u\left[\cos(\lambda+\theta)P_{2i-1,2i} - \cos\theta(c^+_{2i-1}c_{2i} + c^+_{2i}c_{2i-1})\right],\\
\overset{\vee}{\check{R}}^{\iota_2}_{2i-1,2i}(u) &= \check{R}_{2i-1,2i}(-\theta) + u\left[\cos(\lambda-\theta)P_{2i-1,2i} - \cos\eta(c^+_{2i-1}c_{2i} + c^+_{2i}c_{2i-1})\right], \quad (33)
\end{aligned}$$

where $P_{i,i+1} = n_i n_{i+1} + (1-n_i)(1-n_{i+1})$ and $\check{R}_{2i-1,2i}(\theta)$ defined by formula (28) with

$$a(\theta) = \sin(\lambda+\theta), \qquad b(\theta) = \sin\theta, \qquad c(\theta) = \sin\lambda. \quad (34)$$

It is easy to see in the Figure 4 that, although the $\overset{\vee}{\check{R}}_{2i-1,2i}$-operators are not unity at $u=0$, the Hamiltonian will still be local. The scattering due to $\overset{\vee}{\check{R}}_{2i-1,2i}$-operators will induce an interaction involving at most 4 fermions. For example, the excitation Hamiltonian H_{23} for the fermions 2 and 3 (see Figure 4) due to scatterings in $\overset{\vee}{\check{R}}_{12}(\theta)$ and $\overset{\vee}{\check{R}}_{34}(\theta)$ may create the interaction between 1, 2, 3, 4 states only.

Now, by inserting the expressions (33) and (34) for $\overset{\vee}{\check{R}}^{\iota_1}, \overset{\vee}{\check{R}}^{\iota_2}, \check{R}$ and $\check{R}^{\iota_1\iota_2}$ into the monodromy matrix $T(u) = T_0(u)T_1(u)$, in the linear approximation of u we obtain

$$T(u) \approx (\sin^2\lambda\sin(\lambda+\theta)\sin(\lambda-\theta))^N \left(1 + u\sum_{j=1}^{N} H_j\right), \quad (35)$$

with

$$\begin{aligned}
&\sin\lambda\sin(\lambda+\theta)\sin(\lambda-\theta)H_j =\\
&\quad +(-1)^j\sin\theta\sin 2\lambda(1-n_{j-1}-n_{j+2})(c^+_j c_{j+1} - c^+_{j+1}c_j)\\
&\quad +(-1)^{j+1}\sin^2\theta\cos\lambda\left[n_j n_{j+2} + (1-n_j)(1-n_{j+2})\right]\\
&\quad +\sin\theta\left[\left(\sin(\lambda+(-1)^j\theta)(1-n_{j+1}) + \sin(\lambda-(-1)^j\theta)n_{j+1}\right)c^+_j c_{j+2}\right.\\
&\quad \left. -\left(\sin(\lambda-(-1)^j\theta)(1-n_{j+1}) + \sin(\lambda+(-1)^j\theta)n_{j+1}\right)c^+_{j+2}c_j\right]. \quad (36)
\end{aligned}$$

In the limit $\lambda = \pi/2$, which corresponds to XX-model, we will have two types of independent free fermions, hopping separately at the even and odd sites of the chain with the Hamiltonian [6]

$$H_j = \tan^2\theta\left[c^+_j c_{j+2} - c^+_{j+2}c_j\right]. \quad (37)$$

It is convenient now to write the Hamiltonian (36) in a ladder form represented graphically as in Figure 5.

[6] It seems to us that the model considered in [10] do not reproduces the free fermionic limit at $\lambda = \pi/2$, as in case presented here.

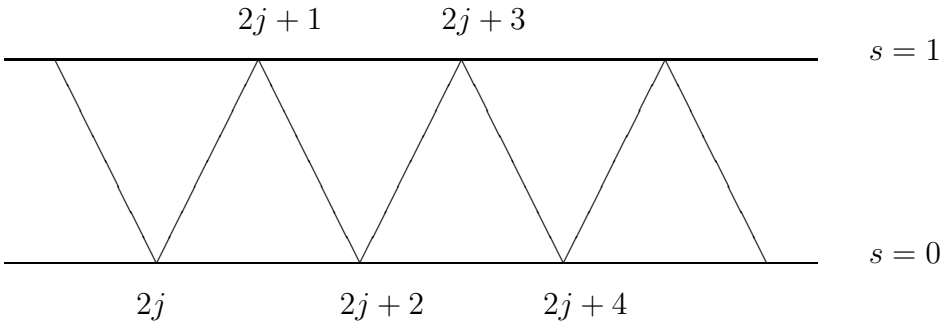

Figure 5: Zig-zag ladder chain

Let us consider the even $(2j)$ and the odd $(2j+1)$ points of the chain as a sites (j) of two different chains labeled by $s = 0$ and 1 correspondingly. The Fermi fields will be marked now as

$$c_{j,s} = c_{2j+s}, \qquad s = 0, 1 \tag{38}$$

With these notations it is straightforward to obtain from the expression (36) the following ladder Hamiltonian

$$\begin{aligned}
&\sin(\lambda+\theta)\sin(\lambda-\theta)\sin(\lambda)H_{j,s} = \\
&\quad (-1)^s \sin\theta \sin 2\lambda \left(c^+_{j,s}c_{j,s+1} - c^+_{j,s+1}c_{j,s}\right)(\bar{n}_{j-1,s+1} - n_{j+1,s}) \\
&\quad +(-1)^{s+1}\sin^2\theta\cos\lambda\left[n_{j,s}n_{j+1,s} + (1-n_{j,s})(1-n_{j+1,s})\right] \\
&\quad +\sin\theta\left[c^+_{j,s}c_{j+1,s}\left(\sin(\lambda+(-1)^s\theta)(1-n_{j,s+1}) + \sin(\lambda-(-1)^s\theta)n_{j,s+1}\right)\right. \\
&\quad \left. -c^+_{j+1,s}c_{j,s}\left(\sin(\lambda+(-1)^s\theta)n_{j,s+1} + \sin(\lambda-(-1)^s\theta)(1-n_{j,s+1})\right)\right].
\end{aligned} \tag{39}$$

Hence, we have obtained a simple integrable ladder Hamiltonian, which has a free fermionic limit.

We can also write down the Hamiltonian of the model in a spin language, calculated by use of the ordinary representation for $\check{R}^{a'b'}_{ab}$ as

$$\check{R}^{a'b'}_{ab} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

In ladder form the result looks as follows

$$\begin{aligned}
\sin(\lambda+\theta)\sin(\lambda-\theta)\sin(\lambda)H_j = \\
&\frac{(-1)^s}{2}\sin^2\theta\cos\lambda\left[\sigma^1_{j,s}\sigma^1_{j+1,s}+\sigma^2_{j,s}\sigma^2_{j+1,s}-\sigma^3_{j,s}\sigma^3_{j+1,s}\right] \\
+ \quad & \sin\theta\sin\lambda\left\{\cos\lambda(-1)^s\sigma^3_{j,s}\left(\sigma^+_{j,s+1}\sigma^-_{j+1,s}-\sigma^-_{j,s+1}\sigma^+_{j+1,s}\right)\right. \\
- \quad & (-1)^s\cos\lambda\left(\sigma^+_{j,s}\sigma^-_{j,s+1}-\sigma^-_{j,s}\sigma^+_{j,s+1}\right)\sigma^3_{j+1,s} \\
- \quad & \left.\cos\theta\left(\sigma^+_{j,s}\sigma^3_{j,s+1}\sigma^-_{j+1,s}-\sigma^-_{j,s}\sigma^3_{j,s+1}\sigma^-_{j+1,s}\right)\right\}
\end{aligned} \tag{40}$$

As one can see from this expression, the first term gives XXZ models for the each leg of the ladder with anisotropy parameters $\Delta=-1$ and therefore can be considered as $SU(1,1)$ Heisenberg chains. The other terms, which represents the interaction between chains, are written for the each triangle of the ladder (see Figure 5) and has a topological form

$$\hat{\epsilon}^{abc}\sigma^a_{j,s}\sigma^b_{j,s+1}\sigma^c_{j+1,s}, \tag{41}$$

where $\hat{\epsilon}^{abc}$ is an anisotropic antisymmetric tensor, defined by the coefficients in the formula (40).

4 Algebraic Bethe Ansatz (ABA) solution of the model

The technique of ABA , called also Quantum Inverse Scattering Method (QISM), essentially was developed in the works of Baxter [8] and Faddeevs group [7].

In order to carry out ABA it is convenient to work in conventional (not braid) formalism and use the YBE for R-operators in the form of formulas (17, 18). Let us first define the L-matrix as

$$\begin{aligned}
(L_j(u))^a_{a'} &= \langle a \mid R_{ij} \mid a'\rangle_i = (-1)^{p(a')p(b')}\left(R_{ij}(u)\right)^{ab}_{a'b'}X^{b'}_{jb} \\
&= \begin{pmatrix} a(u)(1-n_j)+b(u)n_j & c(u)c^+_j \\ c(u)c_j & -a(u)n_j+b(u)(1-n_j)\end{pmatrix},
\end{aligned}$$

which is a matrix in the horizontal auxiliary space with operator value entities and acting on quantum space V_j. The matrix elements between auxiliary states $\mid a'\rangle$ and $\langle a \mid$ of the monodromy operators $T_{0,1}(\lambda,u)$ defined in (14)

$$(T_s(u))^{a'}_a = \langle a' \mid T_s(u) \mid a\rangle = \begin{pmatrix} A_s(u) & B_s(u) \\ C_s(u) & D_s(u)\end{pmatrix}, \qquad s=0,1$$

can be expressed as a product of the L-matrices as follows

$$\begin{aligned}
(T_1)^{a_0}_{a_N} &= (L_1)^{a_0}_{a_1}(u)\,(L^{\iota_2}_2)^{a_1}_{a_2}(\bar{u})\ldots..\,(L^{\iota_2}_1)^{a_{2N-1}}_{a_{2N}}(\bar{u}), \\
(T_0)^{a_0}_{a_N} &= (L^{\iota_1}_1)^{a_0}_{a_1}(\bar{u})\,(L^{\iota_1\iota_2}_2)^{a_1}_{a_2}(u)\ldots..\,(L^{\iota_1\iota_2}_1)^{a_{2N-1}}_{a_{2N}}(u).
\end{aligned} \tag{42}$$

By use of equations (17, 18) one can obtain the graded YBE for the monodromy matrices $T_s, \qquad s = 0, 1$ as follows

$$(-1)^{(P(b')+P(b''))P(a'')} R^{ab}_{a'b'}(u,v)\,(T_0)^{a'}_{a''}(u)\,(T_1)^{b'}_{b''}(v)$$
$$= (-1)^{P(b')(P(a')+P(a))}\,(T_0)^{b}_{b'}(v)\,(T_1)^{a}_{a'}(u)\widetilde{R}^{a'b'}_{a''b''}(u,v),$$

$$(-1)^{(P(b')+P(b''))P(a'')} \widetilde{R}^{ab}_{a'b'}(u,v)\,(T_1)^{a'}_{a''}(u)\,(T_0)^{b'}_{b''}(v)$$
$$= (-1)^{P(b')(P(a')+P(a))}\,(T_1)^{b}_{b'}(v)\,(T_0)^{a}_{a'}(u)R^{a'b'}_{a''b''}(u,v). \qquad (43)$$

Now, following the procedure of ABA, let us define empty fermionic state

$$\mid \Omega\rangle = \prod_{i=1}^{2N} \mid 0\rangle_i \qquad (44)$$

as a test vacuum of the model and demonstrate that it is the eigenstate of the transfer matrix $\tau(\lambda,u) = \tau_1(\lambda,u)\tau_0(\lambda,u) = strT(\lambda,u)$.

The expression (42) for L-matrix shows that his action on $\mid 0\rangle_i$ produces upper triangular matrix, therefore the action of the monodromy matrix $(T_s)^{a'}_{a}$ on $\mid \Omega\rangle$, defined by the formulas (42) and (42), will also have upper triangular form and can be calculated easily as

$$\begin{aligned}
(T_1(u)) \mid \Omega\rangle &= \begin{pmatrix} [a(u)a^{\iota_2}(\bar{u})]^N & B_1(u) \\ 0 & [b(u)b^{\iota_2}(\bar{u})]^N \end{pmatrix} \mid \Omega\rangle, \\
(T_0(u)) \mid \Omega\rangle &= \begin{pmatrix} [a^{\iota_1}(\bar{u})a^{\iota_1\iota_2}(u)]^N & B_0(u) \\ 0 & [b^{\iota_1}(\bar{u})b^{\iota_1\iota_2}(u)]^N \end{pmatrix} \mid \Omega\rangle.
\end{aligned}$$

Now it is obvious, that $\mid \Omega\rangle$ is eigenstate of $\tau(\lambda,u)$ with eigenvalue

$$\begin{aligned}
\nu(u) &= \nu_1(u)\nu_0(u), \\
\nu_0(u) &= [a^{\iota_1}(\bar{u})a^{\iota_1\iota_2}(u)]^N - [b^{\iota_1}(\bar{u})b^{\iota_1\iota_2}(u)]^N, \\
\nu_1(u) &= [a(u)a^{\iota_2}(\bar{u})]^N - [b(u)b^{\iota_2}(\bar{u})]^N. \qquad (45)
\end{aligned}$$

One can see from the expressions (45) that the operators $C_s(u)$ act on $\mid \Omega\rangle$ as the annihilation operators, while $B_s(u)$ act as the creation operators. That is why it is meaningful to look for states

$$\mid v_1, v_2, ...v_n\rangle_0 = B_1(v_1)B_0(v_2)...B_x(v_n) \mid \Omega\rangle_x, \qquad x = n(mod 2), \qquad (46)$$

as an n-particle eigenstates of $\tau(u)$ with spectral parameters $v_i,\ i = 1, ...n$. In order to check whether this is true we do not need to have an exact form of operators $B_s(u)$, we

need only to know the algebra of operators $A_s(u), D_s(u)$ and $B_s(u)$, which can be found from the YBE (43) as follows

$$A_0(u)B_1(v) = -\frac{a(v-u)}{b(v-u)}B_0(v)A_1(u) + \frac{c(v-u)}{b(v-u)}B_0(u)A_1(v) \quad (47)$$

$$A_1(u)B_0(v) = +\frac{a(v-u)}{b(v-u)}B_1(v)A_0(u) - \frac{c(v-u)}{b(v-u)}B_1(u)A_0(v) \quad (48)$$

and

$$D_0(u)B_1(v) = +\frac{a(u-v)}{b(u-v)}B_0(v)D_1(u) + \frac{c(u-v)}{b(u-v)}B_0(u)D_1(v) \quad (49)$$

$$D_1(u)B_0(v) = -\frac{a(u-v)}{b(u-v)}B_1(v)D_0(u) - \frac{c(u-v)}{b(u-v)}B_1(u)D_0(v). \quad (50)$$

The first terms in the right hand side of equations (47) and (49) are so called "wanted" terms and they are producing the eigenvalues $\nu(u, v_1, v_2, ..v_n)$ of the state $\mid v_1, v_2, ...v_n\rangle_0$ as

$$\begin{aligned}
&\nu(u, v_1, v_2, ..v_n) = \nu_1(u, v_1, v_2, ..v_n)\nu_0(u, v_1, v_2, ..v_n),\\
&\nu_0(u, v_1, \cdots, v_n) = \frac{(-1)^{\frac{n}{2}}}{\prod_{i=1}^{n} b(v_i - u)} \cdot\\
&\qquad \cdot \left\{ \prod_{i=1}^{n} a(v_i - u)[a^{\iota_1}(\bar{u})a^{\iota_1\iota_2}(u)]^N - \prod_{i=1}^{n} a(u - v_i)[b^{\iota_1}(\bar{u})b^{\iota_1\iota_2}(u)]^N \right\},\\
&\nu_1(u, v_1, \cdots, v_n) = \frac{(-1)^{\frac{n}{2}}}{\prod_{i=1}^{n} b(v_i - \bar{u})} \cdot\\
&\qquad \cdot \left\{ \prod_{i=1}^{n} a(v_i - \bar{u})[a(u)a^{\iota_2}(\bar{u})]^N - \prod_{i=1}^{n} a(\bar{u} - v_i)[b(u)b^{\iota_2}(\bar{u})]^N \right\}. \quad (51)
\end{aligned}$$

But in order the state $\mid v_1, v_2, ...v_n\rangle$ to be an eigenstate of $\tau(u)$ we need the cancellation of so called "unwanted" terms, produced by the second terms in the right hand side of equations (47) and (49). This gives us the restrictions on the spectral parameters $v_1, v_2, ...v_n$ in a form of Bethe Equations (BE)

$$\left[\frac{a(v_j)a^{\iota_2}(\bar{v}_j)}{b(v_j)b^{\iota_2}(\bar{v}_j)}\right]^N = -\prod_{i\neq j}^{n} \frac{a(v_j - v_i)}{a(v_i - v_j)}, \qquad j = 1, ...n. \quad (52)$$

These equations are similar, but differ by some sign factors from the Bethe equations obtained for the model defined in [10] caused by the alternating anisotropy model parameter $\pm\Delta$ along the chain.

The calculation of the eigenvalues of the Hamiltonian (39) is straightforward by taking of logarithmic derivatives of (51) at $u = 0$, which gives us

$$\begin{aligned} E &= \sin\lambda \sum_{i=1}^{n} \left(\frac{1}{\sin v_i \sin(\lambda + v_i)} - \frac{1}{\sin(v_i - \theta)\sin(\lambda + v_i - \theta)} \right) \\ &+ N \frac{\sin 2\theta}{\sin(\lambda+\theta)\sin(\lambda-\theta)}. \end{aligned} \tag{53}$$

5 Open chain Hamiltonian and boundary terms of the model in free fermionic case ($\lambda = \pi/2$)

In this section we follow the approach and the equations of [16, 17, 18, 19] to construct the reflection $\mathcal{K}$-matrix for our model in free fermionic case and calculate integrable boundary terms for the Hamiltonian.

In ordinary integrable models one defines the double row transfer matrix as

$$\begin{aligned} \tau(u) &= \mathrm{tr}_0 \left(K_0^+(u) T(u) K_0^-(u) T(-u)^{-1} \right) \\ &= \zeta(u)^{-L} \mathrm{tr}_0 \left(K_0^+(u) \check{R}_{L0}(u) \check{R}_{L-1,L}(u) \cdots \check{R}_{23}(u) \check{R}_{12}(u) \right. \\ &\quad \left. \times K_1^-(u) \check{R}_{12}(u) \check{R}_{23}(u) \cdots \check{R}_{L-1,L}(u) \check{R}_{L0}(u) \right) , \end{aligned} \tag{54}$$

where N is the number of chain sites and $\zeta(u)$ is defined by the unitarity condition

$$\check{R}(u)\check{R}(-u) = \zeta(u) = \cos^2(u) \tag{55}$$

for the XX-model.

The reflection matrices $K(u)$ (for the right side) and $K^+(u)$ (for the left side) have to obey the reflection equations, the analogues of YBE on the boundaries, in order to ensure the commutativity of the double row transfer matrix (54) for different spectral parameters. This, together with the YBE equation, is a sufficient condition for integrability. The reflection equations read

$$\check{R}_{12}(u-v) K_2^-(u) \check{R}_{12}(u+v) K_2^-(v) = K_2^-(v) \check{R}_{12}(u+v) K_2^-(u) \check{R}_{21}(u-v) \tag{56}$$

and

$$\begin{aligned} &\check{R}_{12}(-u+v) K_1^+(u) \check{R}_{12}(-u-v+\pi) K_1^+(v) = \\ &\qquad = K_1^+(v) \check{R}_{12}(-u-v+\pi) K_1^+(u) \check{R}_{12}(-u+v) \, . \end{aligned} \tag{57}$$

After the fermionization (4),(5),(8), the solutions of the equations (56) and (57) have the following form

$$K(u) = \sin(\xi + u) n + \sin(\xi - u)(1 - n) \tag{58}$$

and

$$K^+(u) = -\sin(\xi^+ - u)n + \sin(\xi^+ + u)(1-n), \tag{59}$$

where ξ and ξ^+ are arbitrary parameters.

In the present model, since we already have a two-row transfer matrix in the closed chain case, the open chain transfer matrix contains four rows, with two- normal and two-backward directions.

Let us define

$$\begin{aligned} \tau(u) = \tau_0(u)\tau_1(u) &= \zeta(u)^{-N} tr_0\left[K_0^+(\theta-u)T_0(\theta-u)K_0(\theta-u)T_0)^{-1}(-\theta+u)\right] \times \\ &\times \; tr_1\left[K_1^+(u)T_1^{\iota_1}(u)K_1(u)T_1^{\iota_1})^{-1}(-u)\right], \end{aligned} \tag{60}$$

where $K_0(u)$, $K_1(u)$, $K_0^+(u)$ and $K_1^+(u)$ are defined by equations (58),(59), while $T_0(u)$ and $T_1(u)$ are defined by (29) after the shift of the spectral parameter.

Let us emphasize here, that because the boundaries of the $T_0(u)$ and $T_1(u)$ in (29) can be defined in a various ways, namely, the beginning and the end of the chain can be translated by one R-matrix, we will have a variety of boundary transfer matrices. We now consider only one possible solution.

The boundary transfer matrix $\tau(u)$ can be expressed graphically as in Figure 6.

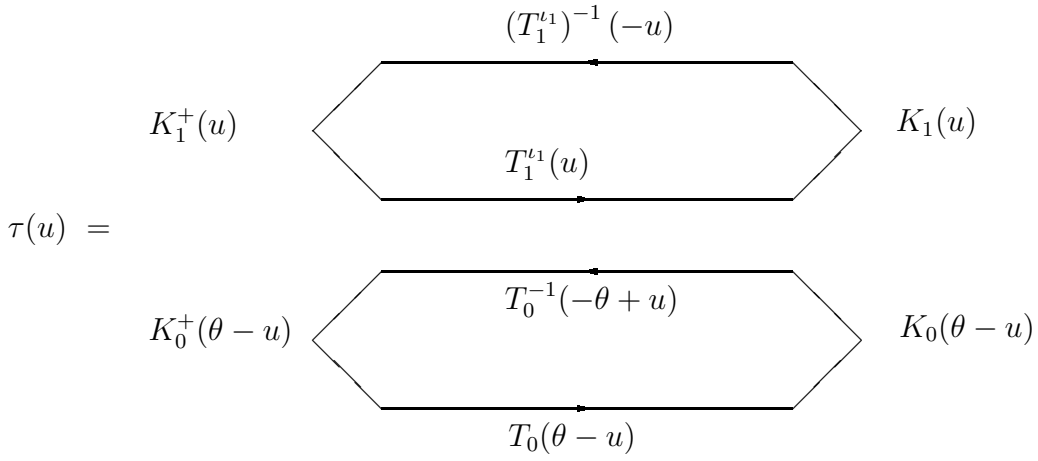

Figure 6: Boundary transfer matrix

This way of writing the boundary transfer matrix $\tau(u)$ is not convenient for the calculation of the Hamiltonian, particularly for the bulk part, but it is convenient to analyze its commutation properties for different spectral parameters. It is clear that the boundary YBEs (reflection equations) are the same as (56) and (57) for the ordinary case. The expressions (58) and (59), after the shift of the spectral parameter $u \to \theta - u$ for the 0 row will ensure the commutativity of the boundary transfer matrices (60). Moreover, by use of YBE (17) and (18) we can translate the $T_0^{-1}(-\theta+u)$ row in Figure 6 to very top

by commuting with $T_1^{\iota_1}(u)$ and $(T_1^{\iota_1})^{-1}(-u)$ rows and bring it to the form, convenient for the calculation of the Hamiltonian:

$$\tau(u) = tr\left[\mathcal{K}^+(u)T_0(\theta-u)T_1(u)\mathcal{K}(u)T_1^{-1}(-u)T_0^{-1}(-\theta+u)\right] \ . \tag{61}$$

Here the effective reflection matrices $\mathcal{K}^+(u), \mathcal{K}(u)$ have been introduced and are defined as follows

$$\begin{aligned} \mathcal{K}(u) &= \check{R}(2u+\theta)K_1(u)\check{R}(\theta)K_0(\theta-u), \\ \mathcal{K}^+(u) &= -\check{R}(\theta-2u)K_1^+(u)\check{R}(-\theta)K_0^+(\theta-u). \end{aligned} \tag{62}$$

It is clear, that the boundary transfer matrix $\tau(u)$ can be represented graphically as in Figure 7.

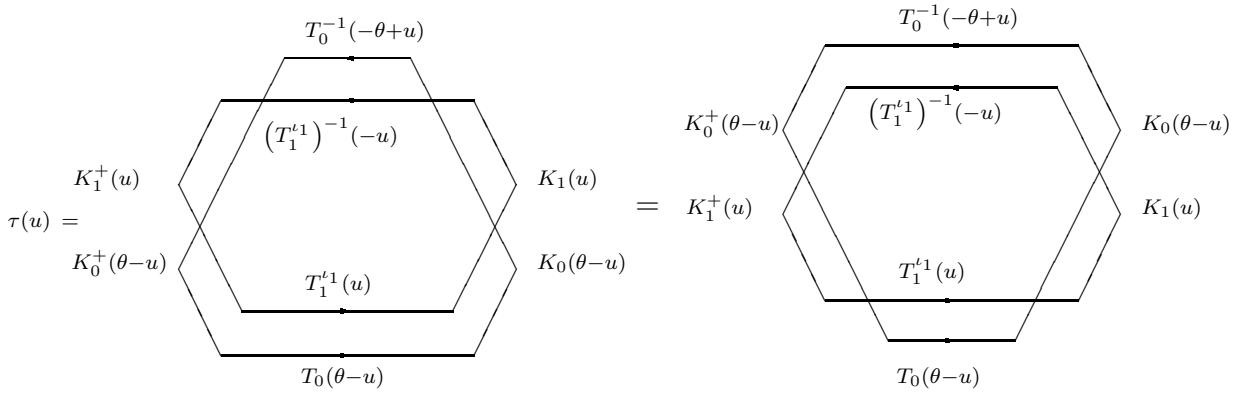

Figure 7: Identity relation for boundary transfer matrix

The equation in Figure 7 is a consequence of the reflection equations (56) and (57).

Now again, in order to calculate the boundary terms, we should input into the expression of the boundary transfer matrix (61), as in the case of the bulk Hamiltonian, the linear expansions of the operators $\check{R}, \check{\check{R}}$, defined by the formulas (33 - 34) and the following linear expansions of the reflection K-matrices around $u = 0$

$$\begin{aligned} K_i(u) &= \sin\xi + u\cos\xi\left[n_i - (1-n_i)\right] \ , \\ K_i(\theta-u) &= \left[\sin(\xi+\theta)n_i + \sin(\xi-\theta)(1-n_i)\right] + \\ &+ u\left[-\cos(\xi+\theta)n_i + \cos(\xi-\theta)(1-n_i)\right] \end{aligned} \tag{63}$$

and extract the terms linear in v. After some calculations one can obtain

$$H = \sum_{j=1}^{N} 2H_j + H_R + H_L, \tag{64}$$

where H_j is the bulk Hamiltonian, which have been calculated in the previous section and given by the formula (37), while the right -H_R and the left- H_L boundary terms have the following expressions

$$\begin{aligned} H_R &= \tan\theta\left(\left[\frac{\sin(\xi-\theta)}{\sin(\xi+\theta)}+1\right]c_N^+c_{N-2}-\left[\frac{\sin(\xi+\theta)}{\sin(\xi-\theta)}+1\right]c_{N-2}^+c_N\right) \\ &+ \frac{1}{\cos\theta}\left(\left[\frac{\sin(\xi-\theta)}{\sin(\xi+\theta)}-1-2\tan\theta\right]c_{N-1}^+c_N+\left[\frac{\sin(\xi+\theta)}{\sin(\xi-\theta)}-1+2\tan\theta\right]c_N^+c_{N-1}\right) \\ &+ \left[\frac{\cot\xi}{\cos^2\theta}-\cot(\xi+\theta)\right]n_N+-\left[\frac{\cot\xi}{\cos^2\theta}-\cot(\xi-\theta)\right](1-n_N) \\ &- \tan^2\theta\cot\xi n_{N-1}+\tan^2\theta\cot\xi(1-n_{N-1})-2\tan\theta, \\ H_L &= -4\frac{\sin\theta}{\sin\xi\cos 2\theta}=const \end{aligned} \tag{65}$$

It is important to mention that, besides the translation of the boundary of the chain, there is another possibility for a modification of the boundary transfer matrix (60). We can take the return paths $T_0^{-1}(-\theta+u)$ and $(T_1^{\iota_1})^{-1}(-u)$ not above the direct chain, as it was considered and presented in the Figure 6, but below. Then it is not hard to see that this will interchange the left and right boundary terms (65) in the Hamiltonian. Finally, there are four integrable models with different boundary terms, only one of which being explicitly written here.

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